

FREE PARATOPOLOGICAL GROUPS

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ABSTRACT. In this paper, we study the free paratopological groups $FP(X)$ and $AP(X)$ on a topological space X in the sense of Markov. We prove that $FP(X)$ and $AP(X)$ on an Alexandroff space X are Alexandroff spaces, we introduce simple description of neighborhood bases at the identity for their topologies when the space X is Alexandroff and we give some properties on these neighborhood bases. As applications of this, we give a characterization for $FP(X)$ and $AP(X)$ on an arbitrary space X to be topological groups and we prove that $FP(X)$ is T_0 if the space X is T_0 . In addition, we give a class of a space X , where $FP(X)$ and $AP(X)$ have the inductive limit property.

1. INTRODUCTION

The free paratopological groups $FP(X)$ and $AP(X)$ on a topological space X in the sense of Markov are the free abstract groups $F_a(X)$ and $A_a(X)$ on X with the strongest paratopological group topologies on $F_a(X)$ and $A_a(X)$, respectively, that induce the original topology on X . This paper is adapted from ([5], chapter 3) and it is a study of the topology of free paratopological groups $FP(X)$ and $AP(X)$ on a space X .

In this paper, we prove that the free paratopological groups $FP(X)$ and $AP(X)$ are Alexandroff spaces if and only if the space X is Alexandroff and then we introduce simple neighborhood bases at the identities of $FP(X)$ and $AP(X)$ for their topologies when the space X is Alexandroff. We study some properties of these neighborhood bases and then as applications of this, we study $FP(X)$ and $AP(X)$ in a case where X is a partition space and in another case where X is a T_0 Alexandroff space. In addition, in Theorem 5.3, we characterize the spaces X for which the paratopological groups $FP(X)$ and $AP(X)$ are topological groups and in Theorem 5.8, we prove that $FP(X)$ is T_0 space if the space X is T_0 . These last two results were found independently by the

2010 *Mathematics Subject Classification.* Primary 22A30; secondary 54D10, 54E99, 54H99.

Key words and phrases. topological group, Paratopological group, free paratopological group, Alexandroff space, partition space, neighborhood base at the identity.

author's thesis [5], but later we were informed by Ravsky that similar to these results were found by Pyrch ([10], [11]). Nevertheless we think that our proofs are easier than that the proofs of ([10], [11]). Finally, we give a class of spaces for which the free paratopological groups have the inductive limit property.

2. DEFINITIONS AND PRELIMINARIES

A *paratopological group* is a pair (G, \mathcal{T}) , where G is a group and \mathcal{T} is a topology on G such that the mapping $(x, y) \mapsto xy$ of $G \times G$ into G is continuous. If in addition, the mapping $x \mapsto x^{-1}$ of G into G is continuous, then (G, \mathcal{T}) is a *topological group*.

If (G, \mathcal{T}) is a paratopological group, then simply we denote it by G .

Marin and Romaguera [8] described a complete neighborhood base at the identity of any paratopological group as follows:

Proposition 2.1. *Let G be a group and let \mathcal{N} be a collection of subsets of G , where each member of \mathcal{N} contains the identity element e of G . Then the collection \mathcal{N} is a base at e for a paratopological group topology on G if and only if the following conditions are satisfied:*

- (1) *for all $U, V \in \mathcal{N}$, there exists $W \in \mathcal{N}$ such that $W \subseteq U \cap V$;*
- (2) *for each $U \in \mathcal{N}$, there exists $V \in \mathcal{N}$ such that $V^2 \subseteq U$;*
- (3) *for each $U \in \mathcal{N}$ and for each $x \in U$, there exists $V \in \mathcal{N}$ such that $xV \subseteq U$ and $Vx \subseteq U$; and*
- (4) *for each $U \in \mathcal{N}$ and each $x \in G$, there exists $V \in \mathcal{N}$ such that $xVx^{-1} \subseteq U$.*

Definition 2.2. [3] Let X be a subspace of a paratopological group G . Suppose that

- (1) *the set X generates G algebraically, that is, $\langle X \rangle = G$ and*
- (2) *every continuous mapping $f : X \rightarrow H$ of X to an arbitrary paratopological group H extends to a continuous homomorphism $\hat{f} : G \rightarrow H$.*

Then G is called the *Markov free paratopological group on X* , and is denoted by $FP(X)$.

By substituting “abelian paratopological group” for each occurrence of “paratopological group” above we obtain the definition of the *Markov free abelian paratopological group on X* and we denote it by $AP(X)$.

Remark 2.3. We denote the free topology of $FP(X)$ by \mathcal{T}_{FP} and the free topology of $AP(X)$ by \mathcal{T}_{AP} and we note that the topologies \mathcal{T}_{FP} and \mathcal{T}_{AP} are the strongest paratopological group topologies on the

underlying sets of $FP(X)$ and $AP(X)$, respectively, that induce the original topology on X .

Let X be a set. Then for all $k \in \mathbb{Z}$ we define the subset $Z_k(X) = \{x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n} \in F_a(X) : \sum_{i=1}^n \epsilon_i = k\}$ of the free group $F_a(X)$ and the subset $Z_k^A(X) = \{\epsilon_1 x_1 + \epsilon_2 x_2 + \dots + \epsilon_n x_n \in A_a(X) : \sum_{i=1}^n \epsilon_i = k\}$ of the free abelian group $A_a(X)$. For every $k_1, k_2 \in \mathbb{Z}$, the sets $Z_{k_1}(X)$ and $Z_{k_2}(X)$ are disjoint whenever $k_1 \neq k_2$ and the sets $Z_{k_1}^A(X)$ and $Z_{k_2}^A(X)$ are disjoint whenever $k_1 \neq k_2$. The set $Z_0(X)$ is the smallest normal subgroup containing the set $Z_F = \bigcup_{x \in X} x^{-1}X$ and the set $Z_0^A(X)$ is the smallest subgroup containing the set $Z_A = \bigcup_{x \in X} X - x$.

3. P_α -SPACES

Let X be a topological space and α be an infinite cardinal. Following [12], we say that X is a P_α -space if $\bigcap \mathcal{C}$ is open in X for each family \mathcal{C} of open subsets of X with $|\mathcal{C}| < \alpha$.

Let (X, τ) be a topological space and let α be an infinite cardinal. We define the topology τ_α to be the intersection of all topologies \mathcal{O} on X where $\tau \subseteq \mathcal{O}$ and (X, \mathcal{O}) is a P_α -space. Since the discrete topology on X contains τ and is a P_α -space, τ_α exists and it is easy to see that (X, τ_α) is a P_α -space. We call the topology τ_α the P_α -modification of τ .

Let α be an infinite singular cardinal and let X be a topological space. Then X is a P_{α^+} -space if X is a P_α -space, where α^+ is the successor cardinal of α .

For the remain of this section we assume that α is a fixed infinite cardinal unless we say otherwise.

Theorem 3.1. *Let (X, τ) be a topological space and let α^+ be the infinite successor cardinal of α . Then the collection of all sets which are the intersection of fewer than β open subsets of X is a base for the topology τ_α on X , where $\beta = \alpha$ if α is regular and $\beta = \alpha^+$ if α is singular.*

Proof. Let $\tau = \{U_i\}_{i \in I}$. We show that the collection $\mathcal{B} = \{\bigcap_{d \in D} U_d : D \subseteq I \text{ and } |D| < \beta\}$ of subsets of X is a base for the topology τ_α on X , where β as defined in the statement of the theorem. It is well known that every infinite successor cardinal is regular (for example, see corollary 10.5 of [6]), so in both cases, β is regular.

We show first that \mathcal{B} is a base for some topology τ^* on X . If $x \in X$, then there exists $i_0 \in I$ where $x \in U_{i_0}$ and such that $U_{i_0} \in \mathcal{B}$. Let $B_1, B_2 \in \mathcal{B}$ and let $x \in B_1 \cap B_2$. Assume that $B_1 = \bigcap_{d \in D} U_d$ and

$B_2 = \bigcap_{t \in T} U_t$, where $D, T \subseteq I$ and $|D|, |T| < \beta$. Let $R = D \cup T$. So $|R| < \beta$. Hence $B_3 = \bigcap_{r \in R} U_r \in \mathcal{B}$ and $x \in B_3 \subseteq B_1 \cap B_2$. Therefore, \mathcal{B} is a base for some topology τ^* on X .

We show second that (X, τ^*) is a P_α -space. Let $\tau^* = \{V_j\}_{j \in J}$ and let $M \subseteq J$ where $|M| < \beta$. Then we have

$$\begin{aligned} \bigcap_{m \in M} V_m &= \bigcap_{m \in M} \bigcup_{i \in I_m} B_{m,i} \\ &= \bigcup_{f: M \rightarrow \bigcup_{m \in M} I_m, f(m) \in I_m \forall m \in M} \left(\bigcap_{m \in M} B_{m, f(m)} \right) \in \tau^*, \end{aligned}$$

where I_m is an index set and $B_{m,i} \in \mathcal{B}$ for all $m \in M$ and $i \in I_m$. Thus τ^* contains τ and (X, τ^*) is a P_β -space, which implies that in both cases of β , (X, τ^*) is a P_α -space.

Now let $\hat{\tau}$ be a topology on X containing τ such that $(X, \hat{\tau})$ is a P_α -space. Then in the case where α is regular, we have $\mathcal{B} \subseteq \hat{\tau}$ and in the case where α is singular, by the argument above, we have $(X, \hat{\tau})$ is a P_{α^+} -space, which implies that $\mathcal{B} \subseteq \hat{\tau}$. Thus $\tau^* \subseteq \hat{\tau}$ and hence τ^* is the smallest topology on X containing τ such that (X, τ^*) is a P_α -space. Therefore, $\tau^* = \tau_\alpha$. \square

Next we show that the P_α -modification of the topology of a paratopological group is also a paratopological group topology.

Proposition 3.2. *Let (G, τ) be a paratopological group. Then (G, τ_α) is a paratopological group.*

Proof. Let $g_1, g_2 \in G$ and let $U \in \tau_\alpha$ contain $g_1 g_2$. We show that there exist $U_1, U_2 \in \tau_\alpha$ containing g_1, g_2 , respectively, such that $U_1 U_2 \subseteq U$. Now by Theorem 3.1, there is a set Λ , where $|\Lambda| < \beta$ and β is as in the theorem such that $g_1 g_2 \in \bigcap_{\lambda \in \Lambda} U_\lambda \subseteq U$ where $U_\lambda \in \tau$ for all $\lambda \in \Lambda$. Thus $g_1 g_2 \in U_\lambda$ for all $\lambda \in \Lambda$. Since τ is a paratopological group topology on G , for each $\lambda \in \Lambda$, there are $V(\lambda), W(\lambda) \in \tau$ containing g_1, g_2 , respectively, such that $V(\lambda)W(\lambda) \subseteq U_\lambda$. Let $U_1 = \bigcap_{\lambda \in \Lambda} V(\lambda)$ and $U_2 = \bigcap_{\lambda \in \Lambda} W(\lambda)$. Then $U_1 U_2 \subseteq U_\lambda$ for all $\lambda \in \Lambda$. Hence, $U_1, U_2 \in \tau_\alpha$ and $U_1 U_2 \subseteq \bigcap_{\lambda \in \Lambda} U_\lambda \subseteq U$. Therefore, τ_α is a paratopological group topology on G . \square

Proposition 3.3. *Let (Y, τ_Y) be a subspace of a topological space (X, τ) . Then $\tau_\alpha|_Y = (\tau_Y)_\alpha$.*

Theorem 3.4. *Let X be a topological space. Then the free paratopological group $FP(X)$ on X is a P_α -space if and only if the space X is a P_α -space.*

Proof. \implies : Assume that $FP(X)$ is a P_α -space. Since X is a subspace of $FP(X)$, it is easy to see that X is a P_α -space.

\impliedby : Let τ be the topology of X and let \mathcal{T}_{FP} be the free topology of $FP(X)$. We show that $(\mathcal{T}_{FP})_\alpha = \mathcal{T}_{FP}$. By Proposition 3.2, $(\mathcal{T}_{FP})_\alpha$ is a paratopological group topology on $F_a(X)$ and it is stronger than \mathcal{T}_{FP} . However, \mathcal{T}_{FP} is the free paratopological group topology on $F_a(X)$, which is the strongest paratopological group topology on $F_a(X)$ inducing the original topology τ on X . By Proposition 3.3, we have $(\mathcal{T}_{FP})_\alpha|_X = (\mathcal{T}_{FP}|_X)_\alpha$ and since $(\mathcal{T}_{FP}|_X)_\alpha = (\tau)_\alpha = \tau$, $(\mathcal{T}_{FP})_\alpha$ induces the topology τ of X . Thus we have $(\mathcal{T}_{FP})_\alpha = \mathcal{T}_{FP}$ and therefore, $FP(X)$ is a P_α -space. \square

The same result of Theorem 3.4 is true for $AP(X)$.

A topological space X is said to be *Alexandroff* [1] if the intersection of every family of open subsets of X is open in X .

Our main references for Alexandroff spaces are Kopperman [7] and Arenas [1]. Let X be a topological space and let $x \in X$. Then we define the set

$$U(x) = \bigcap \{U : U \text{ is open in } X \text{ containing } x\}.$$

It is easy to see that a space X is Alexandroff if and only if for each $x \in X$, the set $U(x)$ is open in X . In case when X is an Alexandroff space, we call $U(x)$ the *minimal* open neighborhood of x . It is easy to see that if $y \in U(x)$, then $U(y) \subseteq U(x)$.

A space X is Alexandroff if and only if X is a P_α -space for every infinite cardinal α , so by using Theorem 3.4, we get the next result.

Corollary 3.5. *The free paratopological group $FP(X)$ ($AP(X)$) on a space X is an Alexandroff space if and only if X is an Alexandroff space.*

4. NEIGHBORHOOD BASE AT THE IDENTITY

In this section, we introduce neighborhood bases at the identities of $FP(X)$ and $AP(X)$ on an Alexandroff space X and then we give some properties of these neighborhood bases.

Let G be a group and let H be a subset of G . Then we say that H is a *submonoid* of G if H contains the identity of G and closed under the multiplication in G . If, in addition, H satisfies $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$, then we say that H is a *normal* submonoid of G .

Proposition 4.1. *If H is a normal submonoid of a group G , then $\{H\}$ is a neighborhood base at the identity of G for a paratopological group topology on G .*

Proof. Let H be a normal submonoid of G . Then it is easy to see that $\{H\}$ satisfies the conditions of Proposition 2.1 and then $\{H\}$ is a neighborhood base at the identity of G for a paratopological group topology on G . \square

Proposition 4.2. *Let X be an Alexandroff space. Then the neighborhood base at the identity e (0_A) in $FP(X)$ ($AP(X)$) is a single normal submonoid.*

Proof. By Theorem 3.5, the group $FP(X)$ is an Alexandroff space. Let $U(e)$ be the minimal open neighborhood of the identity element e in $FP(X)$. So $\{U(e)\}$ is a local base at e for the free topology of $FP(X)$. Now there exists a neighborhood V of e in $FP(X)$ such that $V^2 \subseteq U(e)$. Since $U(e) \subseteq V$, $U(e)^2 \subseteq V^2 \subseteq U(e)$. Therefore, $U(e)$ is a submonoid.

Now if $g \in FP(X)$, then by condition (4) of Proposition 2.1, there exists a neighborhood W of e such that $gWg^{-1} \subseteq U(e)$. Since $U(e)$ is the minimal open neighborhood of e , then $U(e) \subseteq W$, which implies that $gU(e)g^{-1} \subseteq gWg^{-1} \subseteq U(e)$. Therefore, $U(e)$ is a normal submonoid of $FP(X)$.

Similarly, we can prove the statement of the theorem for $AP(X)$. \square

Let G be a group. For $S \subseteq G$, let $\langle S \rangle^G$ denote the intersection of all normal submonoids H of G such that $S \subseteq H$. Therefore, $\langle S \rangle^G$ will be the smallest normal submonoid of G containing the set S . We call it the *normal submonoid of G generated by S* and we call the set S the *set of generators* for $\langle S \rangle^G$.

The normal submonoid $\langle S \rangle^G$ is of the form

$$\langle S \rangle^G = \{g_1 s_1 g_1^{-1} g_2 s_2 g_2^{-1} \cdots g_n s_n g_n^{-1} : , g_k \in G, s_k \in S \text{ for all } k = 1, 2, \dots, n, n \in \mathbb{N} \cup \{0\}\},$$

where we use the convention that an empty product is e .

Now we start to introduce a neighborhood base at the identity of $AP(X)$, where X is an Alexandroff space.

Let X be an Alexandroff space and let $U_A = \bigcup_{x \in X} (U(x) - x) \subseteq AP(X)$. Then we define N_A to be the smallest submonoid of $AP(X)$ containing the set U_A . So N_A is of the form

$$N_A = \{y_1 - x_1 + y_2 - x_2 + \cdots + y_n - x_n : x_i \in X, y_i \in U(x_i) \text{ for all } i = 1, 2, \dots, n, n \in \mathbb{N}\}.$$

Or simply, we write $N_A = \langle U_A \rangle$. Since every submonoid of an abelian group is normal, N_A is normal. However, we will omit the word normal and say submonoid.

Let $\mathcal{N}_A = \{N_A\}$. Since N_A is a submonoid of $AP(X)$, by Proposition 4.1, \mathcal{N}_A is a neighborhood base at 0_A for a paratopological group topology \mathcal{O}_A on $A_a(X)$.

Now we consider the non-abelian case. Let $U'_F = \bigcup_{x \in X} (x^{-1}U(x) \cup U(x)x^{-1}) \subseteq FP(X)$. Then we define N'_F to be the smallest normal submonoid of $FP(X)$ containing the set U'_F .

We claim now that the normal submonoid N'_F is equal to the smallest normal submonoid N_F containing the set $U_F = \bigcup_{x \in X} x^{-1}U(x)$. In fact, it is clear that $N_F \subseteq N'_F$. Now let $w \in U'_F$. If $w = x^{-1}y$, then it is clear that $w \in N_F$ and if $w = yx^{-1}$, then $w = y(x^{-1}y)y^{-1} \in N_F$, which implies that $U'_F \subseteq N_F$. Thus $N'_F \subseteq N_F$ and therefore, $N_F = N'_F$.

The normal submonoid N_F consists exactly of the set of all elements of the form,

$$w = g_1 x_1^{-1} y_1 g_1^{-1} \cdot g_2 x_2^{-1} y_2 g_2^{-1} \cdots g_n x_n^{-1} y_n g_n^{-1}$$

where $n \in \mathbb{N}$, g_1, g_2, \dots, g_n is an arbitrary finite system of elements of $F_a(X)$ and $x_1^{-1}y_1, x_2^{-1}y_2, \dots, x_n^{-1}y_n$ is an arbitrary finite system of elements of U_F .

Define $\mathcal{N}_F = \{N_F\}$. By Proposition 4.1, \mathcal{N}_F is a neighborhood base at e for a paratopological group topology \mathcal{O}_F on the free group $F_a(X)$.

Proposition 4.3. *Let X be an Alexandroff space. Then the topologies \mathcal{O}_F and \mathcal{O}_A induce topologies coarser than the original topology on X .*

Proof. We prove that $\mathcal{O}_F|_X$ induces a topology coarser than the original topology on X . Let $x \in X$ and let $y \in U(x)$. Then $x^{-1}y \in N_F$. This implies that $y \in xN_F \cap X$ and then $U(x) \subseteq xN_F \cap X$. Thus $\mathcal{O}_F|_X$ is coarser than the original topology on X .

An analogous proof can be used for \mathcal{O}_A . □

Theorem 4.4. *Let X be an Alexandroff space. Then \mathcal{N}_F (\mathcal{N}_A) is a neighborhood base at e (0_A) for the topology of $FP(X)$ ($AP(X)$).*

Proof. We prove the theorem for \mathcal{N}_F , since the proof for \mathcal{N}_A is the same. We show first that the topology \mathcal{O}_F is finer than the free topology \mathcal{T}_{FP} of $FP(X)$. Let $\xi: X \rightarrow G$ be a continuous mapping of the space X into an arbitrary paratopological group G . Then ξ extends to a homomorphism $\hat{\xi}: F_a(X) \rightarrow G$. We show that $\hat{\xi}$ is continuous with respect to the topology \mathcal{O}_F . Let V be a neighborhood of $\hat{\xi}(e) = e_G$ in G . Fix $x \in X$. Then $\xi(x)V$ is a neighborhood of $\xi(x)$ in G . Since ξ is

continuous at x , $\xi(U(x)) \subseteq \xi(x)V$ and Since $\hat{\xi}|_X = \xi$, $\hat{\xi}(U(x)) \subseteq \hat{\xi}(x)V$. Because $\hat{\xi}$ is a homomorphism, $\hat{\xi}(x^{-1}U(x)) \subseteq V$. Since x is any point in X , we have

$$(1) \quad \hat{\xi}\left(\bigcup_{x \in X} x^{-1}U(x)\right) \subseteq V.$$

Fix $n \in \mathbb{N}$. Then there exists a neighborhood U of e_G in G such that $U^n \subseteq V$ and also, for all $g \in F_a(X)$, there exists a neighborhood W of e_G in G such that $\hat{\xi}(g)W(\hat{\xi}(g))^{-1} \subseteq U$. Since V is any neighborhood of e_G in G , from (1), we have $\hat{\xi}\left(\bigcup_{x \in X} x^{-1}U(x)\right) \subseteq W$. Fix $g \in F_a(X)$. So we have

$$\hat{\xi}(g)\hat{\xi}\left(\bigcup_{x \in X} x^{-1}U(x)\right)(\hat{\xi}(g))^{-1} \subseteq \hat{\xi}(g)W\hat{\xi}(g)^{-1}.$$

Since $\hat{\xi}$ is a homomorphism,

$$(2) \quad \hat{\xi}\left(\bigcup_{x \in X} gx^{-1}U(x)g^{-1}\right) \subseteq \hat{\xi}(g)W\hat{\xi}(g)^{-1} \subseteq U.$$

Since (2) holds for every $g \in F_a(X)$, we have

$$\hat{\xi}\left(\bigcup_{g \in F_a(X)} \bigcup_{x \in X} gx^{-1}U(x)g^{-1}\right) \subseteq U.$$

Thus we have

$$\hat{\xi}\left(\left(\bigcup_{g \in F_a(X)} \bigcup_{x \in X} gx^{-1}U(x)g^{-1}\right)^n\right) \subseteq U^n \subseteq V.$$

Since n is any element of \mathbb{N} ,

$$\hat{\xi}\left(\bigcup_{n \in \mathbb{N}} \left(\bigcup_{g \in F_a(X)} \bigcup_{x \in X} gx^{-1}U(x)g^{-1}\right)^n\right) \subseteq V$$

Since $N_F = \bigcup_{n \in \mathbb{N}} \left(\bigcup_{g \in F_a(X)} \bigcup_{x \in X} gx^{-1}U(x)g^{-1}\right)^n$, we have $\hat{\xi}(N_F) \subseteq V$. Thus $\hat{\xi}$ is continuous with respect to the topology \mathcal{O}_F and therefore, \mathcal{O}_F is finer than \mathcal{T}_{FP} . By Proposition 4.3, $\mathcal{O}_F|_X$ is coarser than the original topology on X . Since \mathcal{O}_F is finer than \mathcal{T}_{FP} , $\mathcal{O}_F|_X$ induces the original topology on X . Thus we satisfied the conditions of Definition 2.2, which implies that $\mathcal{O}_F = \mathcal{T}_{FP}$. Therefore, \mathcal{N}_F is a neighborhood base at e for the free paratopological group $FP(X)$. \square

Let $FP(X)$ be the free paratopological group on an Alexandroff space X . Then we define the collection $\mathcal{H}_F = \{gN_F : g \in F_a(X)\}$ of left cosets of N_F and we define the collection $\mathcal{H}_A = \{g + N_A : g \in A_a(X)\}$ of (left) cosets of N_A .

Now if $g_1, g_2 \in FP(X)$ such that $g_1 \in g_2 N_F$, then $g_1 N_F \subseteq g_2 N_F N_F = g_2 N_F$.

A similar result is true for the collection \mathcal{H}_A of $AP(X)$.

Let X be a topological space and let $I: X \rightarrow AP(X)$ be the identity mapping of the space X to the free paratopological group $AP(X)$ on X . Then we extend I to the continuous homomorphism mapping $\hat{I}: FP(X) \rightarrow AP(X)$. We call the mapping \hat{I} the *canonical mapping*.

Proposition 4.5. *If X is an Alexandroff space, then $\hat{I}(N_F) = N_A$ and $\hat{I}(Z_0(X)) = Z_0^A(X)$.*

Proof. Let X be an Alexandroff space. Then we have $\hat{I}(U_F) = U_A$ and $\hat{I}(N_F)$ is a submonoid of $AP(X)$ containing the set U_A . We claim that $\hat{I}(N_F) = N_A$. In fact, since $U_A \subseteq \hat{I}(N_F)$ and $\hat{I}(N_F)$ is a submonoid, $N_A \subseteq \hat{I}(N_F)$. It is clear that $\hat{I}(N_F) \subseteq N_A$, therefore, the result follows. Since the normal subgroups $Z_0(X)$ and $Z_0^A(X)$ have Z_F and Z_A , respectively, as sets of generators and $\hat{I}(Z_F) = Z_A$, the same proof can be used to show that $\hat{I}(Z_0(X)) = Z_0^A(X)$. \square

Theorem 4.6. *Let X be an Alexandroff space. Then the following are equivalent.*

- (1) *The space X is indiscrete.*
- (2) *$N_F = Z_0(X)$ in $FP(X)$.*
- (3) *$N_A = Z_0^A(X)$ in $AP(X)$.*

Proof. (1) \Rightarrow (2): Assume that X is indiscrete. Then $U(x) = X$ for all $x \in X$ and so $U_F = Z_F$, where Z_F is the generating set for $Z_0(X)$ (see section 2). Therefore, $N_F = Z_0(X)$.

(2) \Rightarrow (3): Assume that $N_F = Z_0(X)$. Let $\hat{I}: FP(X) \rightarrow AP(X)$ be the canonical mapping. Thus $\hat{I}(N_F) = \hat{I}(Z_0(X))$ and hence by Proposition 4.5, $\hat{I}(N_F) = N_A$ and $\hat{I}(Z_0(X)) = Z_0^A(X)$, so $N_A = Z_0^A(X)$.

(3) \Rightarrow (1): Assume that $N_A = Z_0^A(X)$. Thus $Z_k^A(X)$ is open in $AP(X)$ for each $k \in \mathbb{Z}$. Since $Z_1^A(X) \cap X = X$ and $Z_k^A(X) \cap X = \emptyset$ for all $k \in \mathbb{Z} \setminus \{1\}$, we have X is indiscrete. \square

5. APPLICATIONS

5.1. Free paratopological groups on partition spaces.

Let X be a set and let \mathcal{P} be a partition of X . It is well known in the context of general topology (see for example, Steen [13]) that \mathcal{P} , together with \emptyset , is a base for a topology on X called the *partition topology* generated by \mathcal{P} . This topology is characterized by the fact

that every open set is closed (equivalently, every closed set is open). We call a space X a *partition space* if X has a base which is a partition of X . Clearly, every partition space is an Alexandroff space.

Let X be a partition space. Then it is easy to see that the collection $\{U(x)\}_{x \in X}$ is a partition on X , where $U(x)$ as defined above.

Proposition 5.1. *If X is a partition space, then N_A is a subgroup of $AP(X)$ and N_F is a normal subgroup of $FP(X)$.*

Proof. We show the proposition for N_F . Let X be a partition space and let $w = g_1 x_1^{-1} y_1 g_1^{-1} \cdots g_n x_n^{-1} y_n g_n^{-1} \in N_F$, where $y_i \in U(x_i)$, $g_i \in FP(X)$ for all $i = 1, 2, \dots, n$. Since X is a partition space, $x_i \in U(y_i)$ for all $i = 1, 2, \dots, n$ and hence

$$g_n y_n^{-1} x_n g_n \cdots g_1 y_1^{-1} x_1 g_1 = w^{-1} \in N_F.$$

Therefore, N_F is a normal subgroup.

Essentially the same proof for N_A is valid. \square

Proposition 5.2. *If X is a partition space, then the free paratopological groups $FP(X)$ and $AP(X)$ are partition spaces.*

Proof. Let X be a partition space. Then by Proposition 5.1, the collections \mathcal{H} and \mathcal{H}_A are partitions of $FP(X)$ and $AP(X)$, respectively. Therefore the result follow. \square

Let \mathcal{T}_A be the topology of the subspace X^{-1} of $FP(X)$, where X be any topological space. Then by Theorem 4.2 of [3], the topology \mathcal{T}_A has as an open base the collection $\{C^{-1} : C \text{ closed in } X\}$. In this topology, the intersection of every collection of open subsets is open, and the space $X_A^{-1} = (X^{-1}, \mathcal{T}_A)$ is therefore an Alexandroff space.

Theorem 5.3. *Let X be a topological space. Then the free paratopological group $FP(X)$ on X is a topological group if and only if X is a partition space.*

Proof. \implies : Assume that $FP(X)$ is a topological group. Let U be an open set in X . By the argument above, the topology on the subspace X_A^{-1} of $FP(X)$ has the collection $\{C^{-1} : C \text{ is closed in } X\}$ as a base. Thus $(U^c)^{-1}$ is open in X_A^{-1} . Since the inversion mapping of X^{-1} to X is a homeomorphism, U^c is open in X . So U is closed in X and therefore, X is a partition space.

\impliedby : Assume that X is a partition space. By Proposition 5.1, N_F is a subgroup of $FP(X)$. Therefore, $FP(X)$ is a topological group.

The same proof works for $AP(X)$. \square

5.2. The T_0 separation axiom.

Proposition 5.4. *Let X be an Alexandroff space and let $\hat{I}: FP(X) \rightarrow AP(X)$ be the canonical mapping. Then the space X is T_0 if and only if for each $w \in N_F$ and $w \neq e$ we have $\hat{I}(w) \neq 0_A$.*

Proof. \implies : We prove the contrapositive statement. Suppose that there exists $w \in N_F$ and $w \neq e$ such that $\hat{I}(w) = 0_A$, where

$$w = g_1 x_1^{-1} y_1 g_1^{-1} g_2 x_2^{-1} y_2 g_2^{-1} \cdots g_n x_n^{-1} y_n g_n^{-1} \text{ for some } n \in \mathbb{N}, y_i \neq x_i \text{ and } y_i \in U(x_i) \text{ for all } i = 1, 2, \dots, n.$$

Then we have $\hat{I}(w) = y_1 - x_1 + y_2 - x_2 + \cdots + y_n - x_n = 0_A$. If $n = 1$, then $x_1 = y_1$, which gives a contradiction. Assume that $n > 1$. Since $\hat{I}(w) = 0_A$, for each $i \in A = \{1, 2, \dots, n\}$, there exists $j_i \in A$, where $j_i \neq i$ such that $x_i = y_{j_i}$. Define $\sigma: A \rightarrow A$ by setting $\sigma(i) = j_i$ for all $i \in A$. Clearly σ is a permutation on A . Since any permutation can be written as product of cycles, there are $m \in \mathbb{N}$, where $2 \leq m \leq n$ and distinct $i_1, i_2, \dots, i_m \in A$ such that $\sigma(i_1) = i_2$, $\sigma(i_2) = i_3, \dots, \sigma(i_{m-1}) = i_m, \sigma(i_m) = i_1$ and such that $x_{i_k} = y_{\sigma(i_k)}$ for $k = 1, 2, \dots, m$. Thus $x_{i_1} = y_{i_2}, x_{i_2} = y_{i_3}, \dots, x_{i_{m-1}} = y_{i_m}, x_{i_m} = y_{i_1}$ and hence

$$\begin{aligned} U(y_{i_1}) &\subseteq U(x_{i_1}) = U(y_{i_2}) \subseteq U(x_{i_2}) = U(x_{i_3}) \subseteq \cdots \\ &\subseteq U(x_{i_{m-1}}) = U(y_{i_m}) \subseteq U(x_{i_m}) = U(y_{i_1}), \end{aligned}$$

which implies that

$$U(y_{i_1}) = U(x_{i_1}) = U(y_{i_2}) = U(x_{i_2}) = \cdots = U(x_{i_{m-1}}) = U(y_{i_m}).$$

Thus we can not separate the points $y_{i_1}, x_{i_1}, y_{i_2}, x_{i_2}, \dots, x_{i_{m-1}}, y_{i_m}$. Therefore, X is not a T_0 space.

\Leftarrow : Assume that X is not T_0 . Then there are $x, y \in X$ such that $x \neq y$ and $U(x) = U(y)$, which implies that $x \in U(y)$ and $y \in U(x)$. Hence

$$w = x^{-1} y x y^{-1} = (x^{-1} y)(x(y^{-1} x)x^{-1}) \in N_F$$

and $w \neq e$, but $\hat{I}(w) = 0_A$. Therefore, the space X is T_0 . \square

We note that a corollary of this result is that if X is an Alexandroff T_0 space, then the canonical mapping \hat{I} has the property that $\ker \hat{I} \cap N_F = \{e\}$.

Proposition 5.5. *Let G be a paratopological group. Then G is a T_0 space if and only if for all $a \in G$ such that $a \neq e$, there exists a neighborhood U of e such that either $a \notin U$ or $a^{-1} \notin U$.*

Proof. \implies : Let $a \in G$ such that $a \neq e$. Since G is T_0 , either there exists an open set U such that $e \in U$ and $a \notin U$ or there exists an open set V such that $a \in V$ and $e \notin V$, so defining $U = a^{-1}V$ we have $e \in U$ and $a \notin U$ as required.

\impliedby : Let $a, b \in G$ such that $a \neq b$. Thus $ab^{-1} \neq e$. So by our assumption there exists an open set U containing e such that either $ab^{-1} \notin U$ or $ba^{-1} \notin U$. Thus either $a \notin Ub$ or $b \notin Ua$. Therefore, G is T_0 . \square

Proposition 5.6. *Let X be an Alexandroff space. Then $FP(X)$ is a T_0 space if and only if X is a T_0 space.*

Proof. \implies : Since X is a subspace of $FP(X)$, the result follows.

\impliedby : Assume that X is T_0 . We claim that $FP(X)$ is T_0 . In fact, if $FP(X)$ is not T_0 , then by Proposition 5.5, there exists $w \in FP(X)$, $w \neq e$ such that $w, w^{-1} \in N_F$. Hence by Proposition 5.4, $\hat{I}(w) \neq 0_A$ and by Proposition 4.5, $\hat{I}(w), -\hat{I}(w) \in N_A$, which implies that $\hat{I}(w), -\hat{I}(w)$ are in every neighborhood of 0_A . Once again by Proposition 5.5, $AP(X)$ is not T_0 and so by Proposition 3.4 of [9] (which says that if a space X is T_0 , then $AP(X)$ is T_0), X is not a T_0 space, which gives a contradiction. Therefore, $FP(X)$ is T_0 . \square

Fix $n \in \mathbb{N}$ and let $R_n = \{1, 2, \dots, n\} \subseteq \mathbb{N}$. For $i = 0, 1, \dots, n$, define $R_{n,i} = \{1, 2, \dots, i\}$ and $\tau_n = \{R_{n,i} : i = 0, \dots, n\}$. Then it is easy to see that τ_n is a topology on R_n . Let $m, k \in R_n$, where $m \neq k$ and assume that $m < k$. Then $m \in R_{n,m}$ and $k \notin R_{n,m}$. Therefore, (R_n, τ_n) is a T_0 space.

Proposition 5.7. [5] *Let X be a T_0 space and let x_1, x_2, \dots, x_n be distinct elements of X . Then there exists a continuous mapping $\mu : X \rightarrow R_n$ such that $\mu|_{\{x_1, x_2, \dots, x_n\}}$ is one-to-one.*

Theorem 5.8. *Let X be a topological space. Then the free paratopological group $FP(X)$ on X is T_0 if and only if the space X is T_0 .*

Proof. \implies : Since every subspace of T_0 space is T_0 space, the result follows.

\impliedby : Let $w = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_m^{\epsilon_m} \in FP(X)$ for some $m \in \mathbb{N}$ such that $w \neq e$. Choose indices i_1, i_2, \dots, i_n for some $n \leq m$ such that $x_{i_1}, x_{i_2}, \dots, x_{i_n}$ are the distinct letters among x_1, x_2, \dots, x_m . Then by Proposition 5.7, there exists a continuous mapping $\mu : X \rightarrow R_n$ such that $\mu|_{\{x_{i_1}, x_{i_2}, \dots, x_{i_n}\}}$ is one-to-one, where R_n is the space defined above. Then we extend μ to a continuous homomorphism $\hat{\mu} : FP(X) \rightarrow FP(R_n)$. Since $\mu|_{\{x_{i_1}, \dots, x_{i_n}\}}$ is one-to-one, $\hat{\mu}(w) = [\hat{\mu}(x_1)]^{\epsilon_1} [\hat{\mu}(x_2)]^{\epsilon_2} \cdots [\hat{\mu}(x_n)]^{\epsilon_n} \neq e^*$, where e^* is the identity of $FP(R_n)$. By Proposition 5.6, we have $FP(R_n)$ is a

T_0 space. So there is an open set U in $FP(R_n)$, which contains one of e^* or $\hat{\mu}(w)$ and does not contain the other. Say $e^* \in U$ and $\hat{\mu}(w) \notin U$. Since $\hat{\mu}$ is continuous, $\hat{\mu}^{-1}(U)$ is an open set in $FP(X)$ such that $e \in \hat{\mu}^{-1}(U)$ and $w \notin \hat{\mu}^{-1}(U)$. Similarly for the other case. Therefore, the free paratopological group $FP(X)$ is T_0 . \square

5.3. The inductive limit property in $FP(X)$ and $AP(X)$.

Let X be a topological space. Then the space X is said to be the *inductive limit of a cover* \mathcal{C} if a subset V of X is open whenever $V \cap U$ is open in U for each $U \in \mathcal{C}$.

A parallel result of the next theorem is proved in Proposition 7.4.8 of [2] for the case of free topological groups, now we prove it for the case of free paratopological groups.

Theorem 5.9. *Let X be a T_1 P -space. Then the free paratopological group $FP(X)$ ($AP(X)$) is the inductive limit of the collection $\{FP_n(X) : n \in \mathbb{N}\}$ ($\{AP_n(X) : n \in \mathbb{N}\}$).*

Proof. We prove the statement for $FP(X)$, since the proof for $AP(X)$ is similar. Let C be a subset of $FP(X)$ such that $C \cap FP_n(X)$ is closed in $FP_n(X)$ for all $n \in \mathbb{N}$. By Theorem 4.1.3 of [3], the sets $FP_n(X)$ are closed in $FP(X)$ for all $n \in \mathbb{N}$. Thus the sets $C \cap FP_n(X)$ are closed in $FP(X)$ for all $n \in \mathbb{N}$, which implies that C is a countable union of closed sets in $FP(X)$. Since by Theorem 3.4, $FP(X)$ is a P -space, C is closed in $FP(X)$ and then $FP(X)$ is the inductive limit of the collection $\{FP_n(X) : n \in \mathbb{N}\}$. \square

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